Harmonic oscillator with fluctuating damping parameter

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The multiplicative noise in the equation of motion of an underdamped harmonic oscillator produced by a fluctuating damping parameter has a dramatic effect on the average coordinate of an oscillator. Noise of a sufficiently large strength leads to an instability. In the presence of an external periodic force, the output signal shows a nonmonotonic dependence on the strength and the rate of a color noise (stochastic resonance). Contrary to the case of a random frequency, this effect exists for white noise as well.

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I. INTRODUCTION

A harmonic oscillator subjected to a random force has long been a subject of study. The internal thermal noise appears additively in the oscillator equation of motion while the external noise has been considered multiplicatively randomizing the oscillator frequency. A quantitative investigation of the latter phenomenon started in the remarkable article of Bourret *et al.* [1], and has since been explored further by Russian researchers with application to the electrodynamics of a media with random dielectric constant [2]. An investigation of the response of such a system with nonwhite-noise to an external periodic force demonstrated the existence of stochastic resonance [3].

All these works deal with an additive random force, or with one acting on the oscillator coordinates. We know only one article concerning white noise acting on the velocity, thereby influencing damping, which is related to water waves influenced by a turbulent wind field [4]. However, there are an increasing number of problems where the particles advected by the mean flow passes through the region under study. These include problems of phase transition under shear [5], open flows of liquids [6], Rayleigh-Benard and Taylor-Couette problems in fluid dynamics [7], dendritic growths [8], chemical waves [9], and motion of vortices [10]. The velocity which enters the convective term is subject to fluctuations, i.e., the question arises of a harmonic oscillator with random damping. The same problem appears when one studies the linear stability of nonlinear (say, Duffing or Van der Pol) oscillators.

We consider a forced, underdamped linear oscillator with random damping

$$\frac{d^2x}{dt^2}+\gamma[1+\xi(t)]\frac{dx}{dt}+\omega^2x=a\sin(\Omega t), \tag{1}$$

where the random force $\xi(t)$ is a Gaussian variable with zero mean and white noise correlator

$$\langle\xi(t)\xi(t_1)\rangle=D\delta(t-t_1) \tag{2}$$

or with the exponential correlator

$$\langle\xi(t)\xi(t_1)\rangle=\sigma^2\exp(-\lambda|t-t_1|) \tag{3}$$

which later on will be assumed to be dichotomous noise. In the limit case $\sigma^2\to\infty$ and $\lambda\to\infty$ with $\sigma^2/\lambda=D$, Eq. (3) transforms into Eq. (2).

The generalization of Eq. (1) to an additional additive random force and/or a random frequency should present no problem. In the next two sections we consider separately the solution of the homogeneous equation (1) which is governed by the internal dynamics, and that of the nonhomogeneous equation determining the response to an external field. The full solution of Eq. (1) is given by the sum of these two solutions.

II. FORCE-FREE OSCILLATOR

Let us first consider the free motion of an oscillator, rewriting Eq. (1) for this purpose as

$$L\{x\}=-\gamma\xi\frac{dx}{dt},\quad L\{x\}\equiv\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega^2\right)x. \tag{4}$$

In order to convert the differential equation (1) into an integrodifferential equation we apply, following Ref. [1], the operator L^{-1} to Eq. (4) which gives

$$x=-L^{-1}\left\{\gamma\xi\frac{dx}{dt}\right\}. \tag{5}$$

Using that $L[L^{-1}\{f\}]\equiv f$, one can easily check that the integral operator L^{-1} inverse to the differential operator L defined in Eq. (4) has the following form:

$$L^{-1}\{f\}\equiv\frac{1}{\omega_1}\int_0^t dt_1\exp\left[-\frac{\gamma}{2}(t-t_1)\right]\sin[\omega_1(t-t_1)]f(t_1),$$

$$\omega_1=\sqrt{\omega^2-\frac{\gamma^2}{4}}, \tag{6}$$

i.e.,

$$x(t)=-\frac{\gamma}{\omega_1}\int_0^t dt_1\exp\left[-\frac{\gamma}{2}(t-t_1)\right]\times\sin[\omega_1(t-t_1)]\xi(t_1)\frac{dx}{dt}(t_1) \tag{7}$$

and

$$\frac{dx}{dt}=\frac{\gamma}{\omega_1}\int_0^t dt_1\exp\left[-\frac{\gamma}{2}(t-t_1)\right]\xi(t_1)\frac{dx}{dt}(t_1)\times\left\{\frac{\gamma}{2}\sin[\omega_1(t-t_1)]-\omega_1\cos[\omega_1(t-t_1)]\right\}. \tag{8}$$

On substituting Eq. (8) in the right-hand side of Eq. (4), one obtains

$$\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega^2\right)x(t)$$
$$=-\frac{\gamma^2}{\omega_1}\int_0^t dt_1 \exp\left[-\frac{\gamma}{2}(t-t_1)\right]\xi(t)\xi(t_1)\frac{dx}{dt}(t_1)$$
$$\times\left\{\frac{\gamma}{2}\sin[\omega_1(t-t_1)]-\omega_1\cos[\omega_1(t-t_1)]\right\}. \quad (9)$$

On averaging of Eq. (9), for the noise defined in Eqs. (2) and (3) one can use the simplest version of the splitting of averages [1]

$$\left\langle \xi(t)\xi(t_1)\frac{dx}{dt}(t_1)\right\rangle=\langle\xi(t)\xi(t_1)\rangle\left\langle\frac{dx}{dt}(t_1)\right\rangle. \quad (10)$$

The substitution of Eq. (10) into the averaging equation (9) shows that for white noise (2), one gets

$$\left[\frac{d^2}{dt^2}+\gamma(1-\gamma D)\frac{d}{dt}+\omega^2\right]\langle x\rangle=0, \quad (11)$$

i.e., the presence of white noise in the original equation (1) leads to a decrease of damping if $\gamma D<1$. Moreover, if $\gamma D>1$, i.e., if the noise is sufficiently strong, the effective damping becomes negative, so that the average value of the coordinate x increases in time, which indicates an instability. On the other hand, for the exponentially correlated noise (3) one gets

$$\frac{d^2\langle x\rangle}{dt^2}+\gamma\frac{d\langle x\rangle}{dt}+\frac{\gamma^2\sigma^2}{\omega_1}\int_0^t \exp\left[-\left(\lambda+\frac{\gamma}{2}\right)(t-t_1)\right]$$
$$\times\left\{\frac{\gamma}{2}\sin[\omega_1(t-t_1)]-\omega_1\cos[\omega_1(t-t_1)]\right\}$$
$$\times\frac{d\langle x\rangle}{dt}(t_1)dt_1+\omega^2\langle x\rangle=0. \quad (12)$$

Application of the Laplace transform

$$X(p)=\int_0^\infty\langle x\rangle(t)\exp(-pt)dt \quad (13)$$

to Eq. (12) yields

$$\frac{(p^2+\omega^2+\gamma p)[(p+\lambda)(p+\lambda+\gamma)+\omega^2]-\sigma^2 p\gamma^2(p+\lambda)}{(p+\lambda+\gamma)(p+\lambda)+\omega^2}X(p)$$
$$=\frac{(p+\gamma)[(p+\lambda)(p+\lambda+\gamma)+\omega^2]-\sigma^2\gamma^2(p+\lambda)}{(p+\lambda+\gamma)(p+\lambda)+\omega^2}x(t=0)+\frac{dx}{dt}(t=0). \quad (14)$$

One can check the stability of the solution of Eq. (1), $x=x(t)$, without performing the inverse Laplace transform in Eq. (14). In the absence of a driving force and for zero initial conditions, $x(t=0)=0$, the mean solution $\langle x\rangle$ should relax to zero which means that the solution of the fourth-order polynomial in p in the left-hand side of Eq. (14) must have no roots with a positive real part. According to the Routh-Hurwitz theorem [11], this condition is obeyed for the fourth order equation $\Sigma_{i=0}^{i=4}a_i x^i=0$, if the following relations between coefficients a_i hold:

$$\text{all}\ \ a_i>0,\quad a_1a_4<a_2a_3,\quad a_0a_3^2<a_1a_2a_3-a_1^2a_4. \quad (15)$$

These stability conditions applied to Eq. (14) take the following form:

$$\sigma^2<\min\left\{2\beta+\alpha+(1+\alpha)^2,\quad(1+\alpha^{-1})(\alpha+2\beta),\right.$$
$$\left.\frac{1+\alpha}{2+\alpha}[\alpha+2\beta+2(1+\alpha^2)]\right\},$$

$$(1+\alpha)^2(\delta-\alpha)(\delta+\alpha+2\alpha^2)$$
$$<\{2(1+\alpha)[\delta+(1+\alpha)^2-\sigma^2][(1+\alpha)\delta-\alpha\sigma^2]$$
$$-[(1+\alpha)\delta-\alpha\sigma^2]^2\}, \quad (16)$$

where

$$\alpha=\frac{\lambda}{\gamma},\quad \beta=\frac{\omega^2}{\gamma^2},\quad \delta=\alpha+2\beta. \quad (17)$$

The slightly cumbersome inequalities (16) define the stability conditions in the form of the relations between three parameters σ, α, and β. In the case of white noise ($\sigma^2\to\infty$, $\lambda\to\infty$ with $D=\sigma^2/\lambda=\text{const}$) these inequalities are reduced to the previously obtained condition $D\gamma<1$. In the next section we find the response of an underdamped oscillator to the periodic external field.

III. DRIVEN OSCILLATOR

Equation (1) can be rewritten as two first order differential equations

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=-\gamma y-\gamma\xi y-\omega^2x+a\sin(\Omega t) \quad (18)$$

which, after averaging, take the following form:

$$\frac{d}{dt}\langle x\rangle=\langle y\rangle,$$

$$\frac{d}{dt}\langle y\rangle=-\gamma\langle y\rangle-\gamma\langle\xi y\rangle-\omega^2\langle x\rangle+a\sin(\Omega t). \quad (19)$$

Equation (19) contains a new correlator $\langle\xi y\rangle$ which has to be found separately. To this end, we use the well-known Shapiro-Loginov procedure [12] which for exponentially correlated noise (3) yields

$$\frac{d}{dt}\langle\xi y\rangle=\left\langle\xi\frac{dy}{dt}\right\rangle-\lambda\langle\xi y\rangle. \quad (20)$$

Multiplying the second of Eq. (18) by ξ, one gets after averaging

$$\left\langle\xi\frac{dy}{dt}\right\rangle=-\gamma\langle\xi y\rangle-\gamma\langle\xi^2 y\rangle-\omega^2\langle\xi x\rangle. \quad (21)$$

Equation (21) contains two new correlators $\langle\xi x\rangle$ and $\langle\xi^2 y\rangle$. The former can be easily found using a procedure similar to Eqs. (20), (21), namely,

$$\frac{d}{dt}\langle\xi x\rangle=\left\langle\xi\frac{dx}{dt}\right\rangle-\lambda\langle\xi x\rangle=\langle\xi y\rangle-\lambda\langle\xi x\rangle. \quad (22)$$

To find the higher-order correlator $\langle\xi^2 y\rangle$ one has to use the splitting procedure (10) which gives $\langle\xi^2 y\rangle=\langle\xi^2\rangle\langle y\rangle=\sigma^2\langle y\rangle$. Note that this procedure becomes exact for the special case of the two-state Markov process (dichotomous noise) which is described by the correlator (3) with $\xi=\pm\sigma$. In order to keep our calculation exact, we restrict our attention, similar to the authors of Ref. [1], to the dichotomous noise while for the general case of the colored noise one has to use some approximations. On substituting Eqs. (22) and (21) into Eq. (20) one gets

$$\frac{d}{dt}\langle\xi y\rangle=-\gamma\langle\xi y\rangle-\sigma^2\gamma\langle y\rangle-\omega^2\langle\xi x\rangle-\lambda\langle\xi y\rangle. \quad (23)$$

We thus obtain a system of four equations (19), (22), and (23) for four variables $\langle x\rangle$, $\langle y\rangle$, $\langle\xi x\rangle$, and $\langle\xi y\rangle$. From these equations one can easily find the fourth-order differential equation for $\langle x\rangle$

$$\frac{d^4\langle x\rangle}{dt^4}+2(\lambda+\gamma)\frac{d^3\langle x\rangle}{dt^3}+[2\omega^2+(\lambda+\gamma)^2+\lambda\gamma-\gamma^2\sigma^2]\frac{d^2\langle x\rangle}{dt^2}+[(2\omega^2+\gamma\lambda)(\lambda+\gamma)-\lambda\gamma^2\sigma^2]\frac{d\langle x\rangle}{dt}+\omega^2[\omega^2+\lambda(\lambda+\gamma)]\langle x\rangle=a[\omega^2-\Omega^2+\lambda(\lambda+\gamma)]\sin(\Omega t)+a\Omega(2\lambda+\gamma)\cos(\Omega t). \quad (24)$$

We seek the solution of Eq. (24) in the form

$$\langle x\rangle=\langle x\rangle_0+\langle x\rangle_a, \quad (25)$$

where the output signal $\langle x\rangle_a$ is induced by an external field, $a\sin(\Omega t)$ and $\langle x\rangle_0$ is defined by the internal dynamics. The latter was calculated in the previous section, and its Laplace transform is defined by Eq. (14). Let us write the solution $\langle x\rangle_a$ of the nonhomogeneous Eq. (24) in the form

$$\langle x\rangle_a=A\sin(\Omega t+\phi). \quad (26)$$

Then, one easily finds that

$$A^2=\frac{a^2(f_2^2+\Omega^2\lambda_2^2)}{(\Omega^2\gamma\lambda_2-f_1f_2-\gamma^2\Omega^2\sigma^2)^2+\Omega^2[\lambda_1(\gamma\lambda-2f_1)-\lambda\gamma^2\sigma^2]^2} \quad (27)$$

and

$$\tan\phi=\frac{\Omega\gamma f_2^2+\Omega^3\gamma^2\lambda_2-\Omega\gamma^2\sigma^2(\lambda_1\Omega^2+\lambda\omega^2+\lambda^2\lambda_1)}{f_1(f_2^2+\Omega^2\lambda_2^2)+\Omega^2\gamma^2\sigma^2(f_1+\lambda^2)}, \quad (28)$$

where

$$\lambda_1=\lambda+\gamma,\quad \lambda_2=2\lambda+\gamma,\quad f_1=\Omega^2-\omega^2,\quad f_2=f_1-\lambda\lambda_1. \quad (29)$$

It follows from Eq. (27) that the amplitude of the output signal (26) shows a nonmonotonic dependence on the noise strength σ^2 and the correlation rate λ (stochastic resonance). The amplitude A reaches a maximum at the following value of the noise strength:

$$(\sigma^2)_{\max}=\frac{\gamma(\lambda_2\Omega^2+\lambda_1\lambda^2)-(\Omega^2-\omega^2)(\Omega^2-\omega^2-\lambda\lambda_1)}{\gamma^2(\Omega^2+\lambda^2)}. \quad (30)$$

Dependence of the squared ratio A/a of the amplitude of the response signal to that of the external field on the correlation rate λ for $\omega=\gamma=1$ and different frequencies Ω of the external field is shown in Figs. 1 and 2 for two different noise strength $\sigma^2=1$ and $\sigma^2=5$. These graphs show typical stochastic resonance nonmonotonic behavior for the frequencies Ω close to the resonance frequency $\Omega=\omega=1$. The maxima are more pronounced for larger noise strength.

In the limit case of white Gaussian noise, $\sigma^2\to\infty$, $\lambda\to\infty$ and $\sigma^2/\lambda=D$, Eq. (27) takes the form

$$A=a[(\Omega^2-\omega^2)^2+\gamma^2\Omega^2(1-D\gamma)^2]^{-1/2}. \quad (31)$$

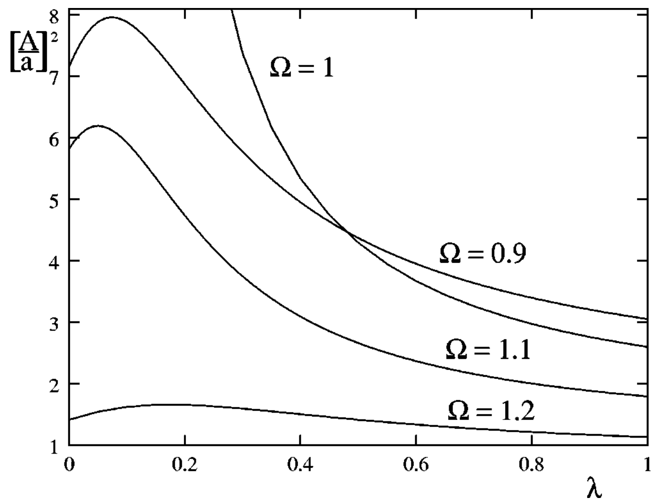

FIG. 1. The dependence of the squared ratio A/a of the amplitude of the response signal to that of the external field on the correlation rate λ for $\gamma=\omega=1$. and $\sigma^2=1$. The curves displayed correspond to different values of the frequency of the external field $\Omega=0.7$, 0.8, 1.0, and 1.2.

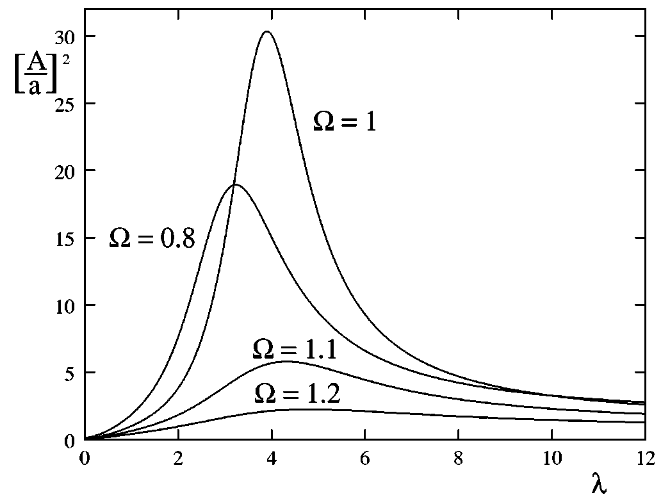

FIG. 2. The same as Fig. 1 for noise strength $\sigma^2=5$ and the frequences $\Omega=0.8$, 1.0, 1.1, and 1.2.

The latter result can be also obtained directly from Eq. (11) with the driving force $a\sin(\Omega t)$ in the right-hand side of this equation. Hence, in the presence of white noise one obtains the "dynamic" resonance slightly renormalized by white noise. The amplitude of the output signal A turns out to be a nonmonotonic function of the noise strength D for white noise as well, reaching its maximum at $D=\gamma^{-1}$. The situation becomes more complicated for color noise (3), where the real "stochastic" resonance occurs. For the resonant frequency $\omega=\Omega$ and $\gamma=1$, the amplitude of the output signal (27) takes the form

$$\frac{A^2}{a^2}=\frac{\lambda^2(\lambda+1)^2+\Omega^2(2\lambda+1)^2}{\Omega^4(2\lambda+1-\sigma^2)^2+\Omega^2\lambda^2(\lambda+1-\sigma^2)^2}, \tag{32}$$

which is a nonmonotonic function of the frequency of an external field Ω, noise strength σ^2, and rate λ. For the special case $\sigma^2=1$, the amplitude increases indefinitely when $\lambda\to 0$.

IV. CONCLUSION

A fluctuating damping parameter, which means the appearance of multiplicative noise in the equation of motion of an underdamped harmonic oscillator, causes an instability for sufficiently large strength of noise ("noisy pump"). A similar effect exists for the case of a random frequency where strong noise may result not only in the well-known instability of the second moments, but also in the instability of the oscillator coordinate (first moment) if the strength of the color noise is sufficiently large. Thus, for an undamped system this strength has to be larger than twice the unperturbed frequency [13]. In our case of a random damping parameter with white noise, such an instability occurs when the noise strength D exceeds an inverse damping parameter γ^{-1}.

Just as in the case of a random frequency, the output signal of an oscillator with a random damping parameter subject to a periodic force shows a nonmonotonic dependence on the strength and the rate of a color noise (stochastic resonance). However, for the random damping parameter, in contrast to the case of a random frequency, this effect exists for white noise as well.

Finally, some terminological clarification should be noted. Sometimes by stochastic resonance one means the nonmonotonic dependence of a signal-to-noise ratio on the noise amplitude. In contrast to this narrow definition, we use the term "stochastic resonance" in the wide sense, meaning the nonmonotonic behavior of the output signal or some function of it (moments, autocorrelation functions or dynamic parameters) on the characteristic of the noise (strength of the noise or the correlation time).

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